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Solitary wave and periodic wave solutions for the thermally forced gravity waves in atmosphere

Zi-Liang Li

Department of Marine Meteorology, Laboratory of Physical Oceanography, Ocean University of China, Songling Road 238, Laoshan District, Qingdao City 266100, People's Republic of China

E-mail: liziliang@ouc.edu.cn.

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Abstract
By introducing a new transformation, a new direct and unified algebraic method for constructing multiple travelling wave solutions of general nonlinear evolution equations is presented and implemented in a computer algebraic system, which extends Fan's direct algebraic method to the case when $r > 4$. The solutions of a first-order nonlinear ordinary differential equation with a higher degree nonlinear term and Fan's direct algebraic method of obtaining exact solutions to nonlinear partial differential equations are applied to the combined KdV–mKdV–GKdV equation, which is derived from a simple incompressible non-hydrostatic Boussinesq equation with the influence of thermal forcing and is applied to investigate internal gravity waves in the atmosphere. As a result, by taking advantage of the new first-order nonlinear ordinary differential equation with a fifth-degree nonlinear term and an eighth-degree nonlinear term, periodic wave solutions associated with the Jacobin elliptic function and the bell and kink profile solitary wave solutions are obtained under the effect of thermal forcing. Most importantly, the mechanism of propagation and generation of the periodic waves and the solitary waves is analysed in detail according to the values of the heating parameter, which show that the effect of heating in atmosphere helps to excite westerly or easterly propagating periodic internal gravity waves and internal solitary waves in atmosphere, which are affected by the local excitation structures in atmosphere. In addition, as an illustrative sample, the properties of the solitary wave solution and Jacobin periodic solution are shown by some figures under the consideration of heating interaction.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Gravity waves in the atmosphere are a subject of broad interest and play a significant role in weather and climate, such as the rainstorm of typhoons, orographic precipitation and atmospheric circulation [1, 2]. One important mechanism of gravity wave production and propagation in the atmosphere is that the airflow response to a transient heat source, which plays a key role in the generation and propagation of internal gravity waves in the atmosphere [3, 4]. One fundamental object of research on wave phenomena is to search for travelling wave solutions [5–11]. Many methods to construct exact solutions of nonlinear wave equations have been established and developed, such as the Bäcklund transformation [12–14], Hirota's bilinear method [15–17], tanh-function method [18–24], extended tanh-function method [25, 26], variational iteration methods [27, 28], collocation method [29–31], Adomian Padé approximation [32], inverse scattering method [13, 33], Darboux transformation [34–38] and so on. Very recently, the F-expansion [39, 40], auxiliary equation [41–43], Fan's sub-equation [44, 45], modified extended Fan's sub-equation methods [46], which are straightforward and effective, were proposed for constructing periodic wave solutions for some nonlinear evolution equations.

The aim of this paper is to consider the first-order nonlinear ordinary differential equation (ODE) with a higher degree nonlinear term

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \epsilon\sqrt{\sum_{j=0}^{r} c_j\varphi^j} \tag{1.1}$$

and with a lower degree nonlinear term

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \epsilon\sqrt{\sum_{j=0}^{s} c_j\phi^j}, \tag{1.2}$$

where $\epsilon = \pm 1, r > 4, s < r$ and $c_j\ (j = 0, 1, \ldots, r)$ are constants.

In fact, when we make a transformation

$$\varphi \to \phi^{(s-2)/(r-2)} \tag{1.3}$$

for equation (1.1) and next specify the coefficients $c_j\ (j = 0, 1, \ldots, r)$, then equation (1.1) is reduced to equation (1.2).

For example, based on Fan's sub-equation method a first-order ODE with a fourth-degree nonlinear term [47–54] is considered, namely

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \epsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}, \tag{1.4}$$

where we choose $s = 4$ in equation (1.3). If we make a transformation

$$\varphi \to \phi^{2/(r-2)}, \tag{1.5}$$

then equation (1.1) is reduced to equation (1.4). If $r = 5$, when we make a transformation $\varphi \to \phi^{2/3}$, then the following first-order ODE with a fifth-degree nonlinear term

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \epsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4 + c_5\varphi^5} \tag{1.6}$$

can be reduced to equation (1.4).

The rest of this paper is organized as follows: in section 2, we derive the mathematical model from the incompressible non-hydrostatic Boussinesq equation, which governs the internal gravity waves in the atmosphere; in section 3, we apply the auxiliary equation method to find various periodic and solitary wave solutions of internal waves, which is used to explain the effect of heat on gravity waves in the atmosphere; in section 4, some conclusions are given.

2. Derivation of the KdV–mKdV–GmKdV equation

Consider the following two-dimensional nonlinear incompressible non-hydrostatic Boussinesq equation [55, 56], which consists of the nonlinear horizontal momentum equation, mass continuity equation and the nonlinear thermodynamic equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \varpi\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} \tag{2.1a}$$

$$\frac{\partial \varpi}{\partial t} + u\frac{\partial \varpi}{\partial x} + \varpi\frac{\partial \varpi}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + B \tag{2.1b}$$

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} - \rho\varpi\frac{N^2}{g} = Q \tag{2.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial \varpi}{\partial z} = 0, \tag{2.1d}$$

where u and ϖ are the velocities, N is the buoyancy frequency per mass unit, p' represents the perturbation pressure and $B = -g\frac{\rho'}{\rho}$ denotes the buoyancy with ρ denoting the density, g denoting the gravitational acceleration and ρ' representing the perturbation density.

Introducing the transformations

$$\begin{aligned} &u(x,z,t) = u(\xi), \qquad \varpi(x,z,t) = \varpi(\xi), \qquad p'(x,z,t) = p(\xi),\\ &\frac{\rho'(x,z,t)}{\rho(x,z,t)} = \pi(\xi), \qquad \xi = kx + nz - wt \end{aligned} \tag{2.2}$$

where w represents an angular frequency, and $\vec{K} = (k, n)$ denotes the wave vector, we can rewrite equation (2.1) as

$$(-w + ku + n\varpi)\frac{\mathrm{d}u}{\mathrm{d}\xi} = -\frac{k}{\rho}\frac{\mathrm{d}p}{\mathrm{d}\xi} \tag{2.3a}$$

$$(-w + ku + n\varpi)\frac{\mathrm{d}\varpi}{\mathrm{d}\xi} = -\frac{n}{\rho}\frac{\mathrm{d}p}{\mathrm{d}\xi} - g\pi \tag{2.3b}$$

$$(-w + ku)\frac{\mathrm{d}\pi}{\mathrm{d}\xi} - \frac{N^2}{g}\varpi = Q \tag{2.3c}$$

$$k\frac{\mathrm{d}u}{\mathrm{d}\xi} + n\frac{\mathrm{d}\varpi}{\mathrm{d}\xi} = 0. \tag{2.3d}$$

Integrating equation (2.3d) with respect to ξ once and taking the integration constant as zero, one can get

$$\varpi = -\frac{ku}{n}. \tag{2.4}$$

Substitution of equation (2.4) into equation (2.3) yields

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \frac{k^2N^2u}{w(k^2+n^2)(w-ku)} - \frac{kngQ}{\rho w(k^2+n^2)(w-ku)} = 0. \tag{2.5}$$

If $u \ll \frac{w}{k}$, then $F(u) = \frac{1}{w-ku}$ can be expanded as

$$F(u) = \frac{1}{w-ku} = \frac{1}{w}\left(1 + \frac{ku}{w} + \frac{k^2u^2}{w^2} + \frac{k^3u^3}{w^3} + \cdots\right). \tag{2.6}$$

Neglecting high power terms in the polynomial of $F(u)$, if we choose that

$$\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\frac{k^2N^2u}{w^2(k^2+n^2)}\left(1+\frac{ku}{w}+\frac{k^2u^2}{w^2}+\frac{k^3u^3}{w^3}\right)-\frac{kngQ}{\rho w^2(k^2+n^2)}\left(1+\frac{ku}{w}+\frac{k^2u^2}{w^2}\right)=0, \quad (2.7)$$

then we can reduce equation (1.5) to

$$\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\frac{k^5N^2u^4}{\omega^5(k^2+n^2)}+\frac{k^4N^2u^3}{\omega^4(k^2+n^2)}+\frac{k^3(N^2\rho\omega-Qgn)u^2}{(k^2+n^2)\rho\omega^4}+\frac{k^2(N^2\rho\omega-Qgn)u}{\omega^3(k^2+n^2)\rho}-\frac{Qgnk}{\omega^2(k^2+n^2)\rho}=0. \quad (2.8)$$

Differentiating equation (2.8) once with respect to ξ leads to

$$\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}+\frac{4k^5N^2u^3}{\omega^5(k^2+n^2)}\frac{\mathrm{d}u}{\mathrm{d}\xi}+\frac{3k^4N^2u^2}{\omega^4(k^2+n^2)}\frac{\mathrm{d}u}{\mathrm{d}\xi}+\frac{2k^3(N^2\rho\omega-Qgn)u}{(k^2+n^2)\rho\omega^4}\frac{\mathrm{d}u}{\mathrm{d}\xi}+\frac{k^2(N^2\rho\omega-Qgn)}{\omega^3(k^2+n^2)\rho}\frac{\mathrm{d}u}{\mathrm{d}\xi}=0, \quad (2.9)$$

which is the ordinary differential equation that the generalized combined KdV–mKdV–GmKdV equation corresponds to.

3. Solutions to the KdV–mKdV–GmKdV equation

Let us now focus our attention on equation (2.9) and introducing the transformations, equation (2.9) is generated as

$$\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}+(c+\alpha u+\beta u^2+\gamma u^3)\frac{\mathrm{d}u}{\mathrm{d}\xi}=0, \quad (3.1)$$

in which

$$\alpha=\frac{2k^3(\rho\omega N^2-ngQ)}{(k^2+n^2)\rho\omega^4},\qquad \beta=\frac{3k^4N^2}{\omega^4(k^2+n^2)},$$
$$\gamma=\frac{4k^5N^2}{\omega^5(k^2+n^2)},\qquad c=\frac{k^2(\rho\omega N^2-ngQ)}{\rho\omega^3(k^2+n^2)}. \quad (3.2)$$

If we let

$$u=\sum_{i=1}^{n}a_i\varphi^i \quad (3.3)$$

with φ satisfying

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\epsilon\sqrt{\sum_{j=0}^{r}c_j\varphi^j}, \quad (3.4)$$

we can conclude that $r=3n+2$ through balancing terms $u^3\frac{\mathrm{d}u}{\mathrm{d}\xi}$ and $\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}$ in equation (3.1). Not losing generality, when we choose $n=1$ and $r=5$, we can set $u=a_0+a_1\varphi$ with φ satisfying

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\epsilon\sqrt{c_0+c_1\varphi+c_2\varphi^2+c_3\varphi^3+c_4\varphi^4+c_5\varphi^5} \quad (3.5)$$

and when we take $n=2$ and $r=8$, we can set $u=a_0+a_1\varphi+a_1\varphi^2$ with φ satisfying

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\epsilon\sqrt{c_0+c_1\varphi+c_2\varphi^2+c_3\varphi^3+c_4\varphi^4+c_5\varphi^5+c_6\varphi^6+c_7\varphi^7+c_8\varphi^8}. \quad (3.6)$$

3.1. The first-order ODE with a fifth-degree nonlinear term

With the aid of Maple9, substitution of equation (3.5) into equation (3.1) shows that the set of algebraic equation possesses the solution

$$c_5 = -\frac{\gamma a_1^3}{10}, \qquad c_4 = -\frac{\gamma a_0 a_1^2}{2} - \frac{\beta a_1^2}{6}, \qquad c_2 = -\alpha a_0 - c - \gamma a_0^3 - \beta a_0^2,$$
$$c_3 = -\frac{2\beta a_0 a_1}{3} - \frac{\alpha a_1}{3} - \gamma a_0^2 a_1, \tag{3.7}$$

where $a_1 \neq 0$, a_0, c and c_0 are arbitrary constants.

Considering the seven parameters a_i $(i = 0, 1)$ and c_j $(j = 0, 1, \ldots, 5)$, especially the γ parameter, we present some types of solutions in the following cases.

Case 1. If $\gamma = 0$, then equation (3.1) is reduced to the well-known combined KdV–mKdV equation, and equation (3.5) takes in the form

$$\frac{d\varphi}{d\xi} = \epsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}. \tag{3.8}$$

If we choose $a_0 = -\frac{\alpha}{2\beta}$ in equation (3.7), then $c_4 \neq 0, c_3 = 0$; we have the following solutions to equation (3.1):

$$u_1 = -\frac{\alpha}{2\beta} + \frac{\sqrt{6(\alpha^2 - 4\beta c)}}{2\beta}\,\mathrm{sech}\left(\sqrt{\frac{\alpha^2 - 4\beta c}{4\beta}}\xi\right), \tag{3.9}$$

where it requires that $\alpha^2 - 4\beta c > 0$ and $\beta > 0$, so these solutions must satisfy the condition

$$n^2 g^2 Q^2 - 2\rho^2 w^2 N^4 + \rho w N^2 n g Q > 0, \qquad N^2 > 0, \tag{3.10}$$

which tells us that in the stable atmosphere $(N^2 > 0)$, heating $(Q > 0)$ helps to generate upward-propagating waves, but cooling $(Q < 0)$ helps to generate downward-propagating waves.

From equation (2.4), we have that the slope of lines of constant phase in the x–z plane is

$$\frac{dz}{dx} = -\frac{k}{n}. \tag{3.11}$$

Considering equations (3.10) and (3.11), it is not difficult to see that in the stable atmosphere $(N^2 > 0)$, heating $(Q > 0)$ helps to generate westerly propagating waves if the slope of lines of constant phase is positive, and heating $(Q > 0)$ helps to generate easterly propagating waves if the slope of lines of constant phase is negative.

However, in the stable atmosphere $(N^2 > 0)$, cooling $(Q < 0)$ helps to generate easterly propagating waves if the slope of lines of constant phase is positive and cooling $(Q < 0)$ helps to generate westerly propagating waves if the slope of lines of constant phase is negative. Figure 1 shows the propagating behaviour in qualitative terms

$$u_2 = -\frac{\alpha}{2\beta} + \frac{1}{2\beta}\sqrt{\frac{4\beta c - \alpha^2}{2}}\tanh\left(\sqrt{\frac{4\beta c - \alpha^2}{8\beta}}\xi\right), \tag{3.12}$$

where it requires that $\alpha^2 - 4\beta c < 0$ and $\beta > 0$; so these solutions must satisfy the condition

$$n^2 g^2 Q^2 - 2\rho^2 w^2 N^4 + \rho w N^2 n g Q < 0, \qquad N^2 > 0, \tag{3.13}$$

which tells us that in the stable atmosphere $(N^2 > 0)$, heating $(Q > 0)$ helps to generate downward-propagating waves, but cooling $(Q < 0)$ helps to generate upward-propagating waves.

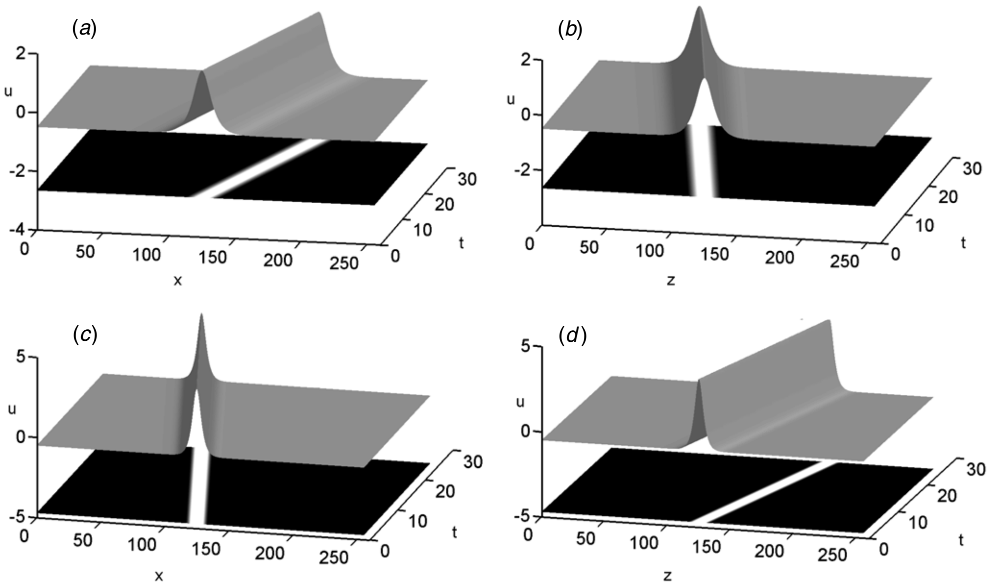

Figure 1. Propagation direction of solitary wave u_1 (equation (3.9), where $N^2 = 0.2, k = 1, n = -1, \omega = 2$) for (*a*) easterly $Q = -0.1$, (*b*) downward $Q = -0.1$, (*c*) westerly $Q = 0.1$, (*d*) upward $Q = 0.1$.

Equations (3.10) and (3.13) show that in the stable atmosphere ($N^2 > 0$), heating ($Q > 0$) helps to generate easterly propagating waves if the slope of lines of constant phase is positive, and heating ($Q > 0$) helps to generate westerly propagating waves if the slope of lines of constant phase is negative.

However, cooling ($Q < 0$) helps to generate westerly propagating waves if the slope of lines of constant phase is positive, and cooling ($Q < 0$) helps to generate easterly propagating waves if the slope of lines of constant phase is negative:

$$u_3 = -\frac{\alpha}{2\beta} - \frac{\sqrt{6}}{\sqrt{-\beta}\xi}, \qquad \alpha^2 - 4\beta c = 0, \qquad \beta < 0, \tag{3.14}$$

where it requires that $n^2g^2Q^2 - 2\rho^2w^2N^4 + \rho w N^2 ngQ = 0$ and $N^2 < 0$. In this case, there are no waves in unstable atmosphere:

$$u_4 = -\frac{\alpha}{2\beta} + \frac{m}{2\beta}\sqrt{\frac{6(\alpha^2 - 4\beta c)}{2m^2 - 1}}cn\left(\sqrt{\frac{\alpha^2 - 4\beta c}{4\beta(2m^2 - 1)}}\xi\right), \tag{3.15}$$

where it requires that $\alpha^2 - 4\beta c > 0$ and $\beta > 0$; so these solutions must satisfy the condition

$$n^2g^2Q^2 - 2\rho^2w^2N^4 + \rho w N^2 ngQ > 0, \qquad N^2 > 0, \tag{3.16}$$

which tells us that in the stable atmosphere ($N^2 > 0$), heating ($Q > 0$) helps to generate upward-propagating periodic waves, but cooling ($Q < 0$) helps to generate downward-propagating periodic waves.

From equations (3.10) and (3.16) we see that in the stable atmosphere ($N^2 > 0$), heating ($Q > 0$) helps to generate westerly propagating periodic waves if the slope of lines of constant phase is positive, and heating ($Q > 0$) helps to generate easterly propagating periodic waves if the slope of lines of constant phase is negative.

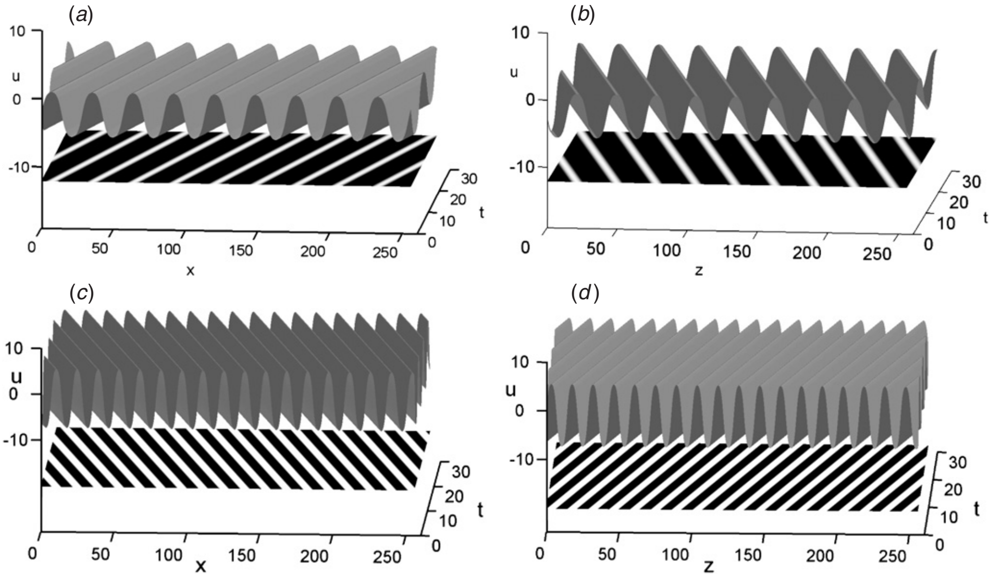

Figure 2. Propagation of periodic wave u_4 (equation (3.15), where $N^2 = 0.2, k = 1, n = -1, \omega = 2$) for (*a*) easterly $Q = -0.1$, (*b*) downward $Q = -0.1$, (*c*) westerly $Q = 0.1$, (*d*) upward $Q = 0.1$.

However, cooling ($Q < 0$) helps to generate easterly propagating periodic waves if the slope of lines of constant phase is positive, and cooling ($Q < 0$) helps to generate westerly propagating periodic waves if the slope of lines of constant phase is negative. The corresponding propagating behaviour is given in figure 2:

$$u_5 = -\frac{\alpha}{2\beta} + \frac{m}{2\beta}\sqrt{\frac{4\beta c - \alpha^2}{m^2+1}}sn\left(\sqrt{\frac{4\beta c - \alpha^2}{4\beta(m^2+1)}}\xi\right), \tag{3.17}$$

where it requires that $\alpha^2 - 4\beta c < 0$ and $\beta > 0$; so these solutions must satisfy the condition

$$n^2g^2Q^2 - 2\rho^2w^2N^4 + \rho w N^2 ngQ < 0, \qquad N^2 > 0, \tag{3.18}$$

which tells us that in the stable atmosphere ($N^2 > 0$), heating ($Q > 0$) helps to generate downward-moving periodic waves, but cooling ($Q < 0$) helps to generate upward-moving periodic waves.

From equations (3.10) and (3.18), it is not difficult to see that in the stable atmosphere ($N^2 > 0$), heating ($Q > 0$) helps to generate easterly moving periodic waves if the slope of lines of constant phase is positive, and heating ($Q > 0$) helps to generate westerly moving periodic waves if the slope of lines of constant phase is negative. However, cooling ($Q < 0$) helps to generate westerly moving periodic waves if the slope of lines of constant phase is positive, and cooling ($Q < 0$) helps to generate easterly moving periodic waves if the slope of lines of constant phase is negative. In addition, when $m \to 1$, u_4 is reduced to u_1 and u_5 is reduced to u_2.

Case 2. If $\gamma = 0$, we set $u = a_0 + \varphi$ and take equation (3.5) in the form

$$\frac{d\varphi}{d\xi} = \epsilon\sqrt{c_0 + c_2\varphi^2 + c_4\varphi^4}, \tag{3.19}$$

which has the Weierstrass elliptic function solution

$$u_6 = -\frac{\alpha}{2\beta} + \sqrt{\frac{\alpha^2 - 4\beta c}{2\beta^2} - \frac{6}{\beta}\wp(\xi; g_2, g_3)}, \tag{3.20}$$

where

$$g_2 = \frac{4}{3}\left(\frac{\alpha^2}{4\beta} - c\right)^2 + \frac{2\beta c_0}{3}, \qquad g_3 = -\frac{8}{27}\left(\frac{3\alpha^2}{4\beta} - c\right)^3 - \frac{2\beta c_0}{9}, \tag{3.21}$$

$$u_7 = -\frac{\alpha}{2\beta} + 2\sqrt{3}\sqrt{\frac{c_0\beta}{12\wp(\xi; g_2, g_3)\beta - (\alpha^2 - 4\beta c)}}, \tag{3.22}$$

in which g_2 and g_3 satisfy equation (3.21)

$$u_8 = -\frac{\alpha}{2\beta} + \sqrt{\frac{12c_0\wp(\xi; g_2, g_3) + 2c_0\left(\frac{\alpha^2-4c\beta}{2\beta} + \mathrm{D}\right)}{12\wp(\xi; g_2, g_3) + \mathrm{D}}}, \tag{3.23}$$

where

$$g_2 = \frac{(-\alpha^2 + 4c\beta)(10\mathrm{D}\beta + 2\alpha^2 - 8c\beta - 11c_0\beta^2)}{96\beta^2}, \tag{3.24}$$

$$g_3 = \frac{7\mathrm{D}\alpha^4}{1152\beta^2} - \frac{7\mathrm{D}\alpha^2 c}{144\beta} + \frac{7\mathrm{D}c^2}{72} + \frac{7c_0\beta\mathrm{D}}{144} + \frac{5\alpha^6}{3456\beta^3} - \frac{5\alpha^4 c}{288\beta^2} + \frac{5\alpha^2c^2}{72\beta} - \frac{5c^3}{54} + \frac{c_0\alpha^2}{192} - \frac{cc_0\beta}{48} \tag{3.25}$$

$$D = \frac{5(-\alpha^2 + 4c\beta)}{8\beta} + \frac{1}{8}\sqrt{\frac{9(-\alpha^2 + 4c\beta)^2}{\beta^2} + 96\beta c_0} \tag{3.26}$$

$$u_9 = -\frac{\alpha}{2\beta} - \frac{1}{12}\frac{\sqrt{c_0}(-24\wp(\xi; g_2, g_3)\beta + 4c\beta - \alpha^2)}{\beta\left(\frac{\partial}{\partial\xi}\wp(\xi; g_2, g_3)\right)}, \tag{3.27}$$

where

$$g_2 = \frac{(4c\beta - \alpha^2)^2}{192\beta^2} - \frac{c_0\beta}{6}, \qquad g_3 = \frac{(4c\beta - \alpha^2)}{864\beta}\left(6c_0\beta + \frac{(4c\beta - \alpha^2)^2}{16\beta^2}\right), \tag{3.28}$$

$$u_{10} = -\frac{\alpha}{2\beta} + \frac{12\sqrt{6}\sqrt{-\beta}\frac{\partial}{\partial\xi}\wp(\xi; g_2, g_3)}{24\wp(\xi; g_2, g_3)\beta - 4c\beta + \alpha^2}, \tag{3.29}$$

in which g_2 and g_3 satisfy equation (3.28). In this case, $\wp(\xi; g_2, g_3)$ in the solutions u_6 to u_{10} is the Weierstrass elliptic function.

Case 3. If $\gamma \neq 0$, when we make a transformation $\varphi \to \varphi^{2/3}$ for equation (3.5), then equation (3.5) is reduced to

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \frac{3}{2}\epsilon\sqrt{c_0\varphi^{2/3} + c_1\varphi^{4/3} + c_2\varphi^2 + c_3\varphi^{8/3} + c_4\varphi^{10/3} + c_5\varphi^4}. \tag{3.30}$$

Suppose that $c_0 = c_1 = c_3 = c_4 = 0$, and thus equation (3.1) possesses a bell-shaped soliton solution

$$u_{11} = -\frac{\beta}{3\gamma} + \left(\frac{10(2\beta^3 + 27c\gamma^2 - 9\alpha\beta\gamma)}{27\gamma^3}\operatorname{sech}^2\left(\frac{\sqrt{-6\beta^3 - 81c\gamma^2 + 27\alpha\beta\gamma}}{6\gamma}\xi\right)\right)^{1/3}, \tag{3.31}$$

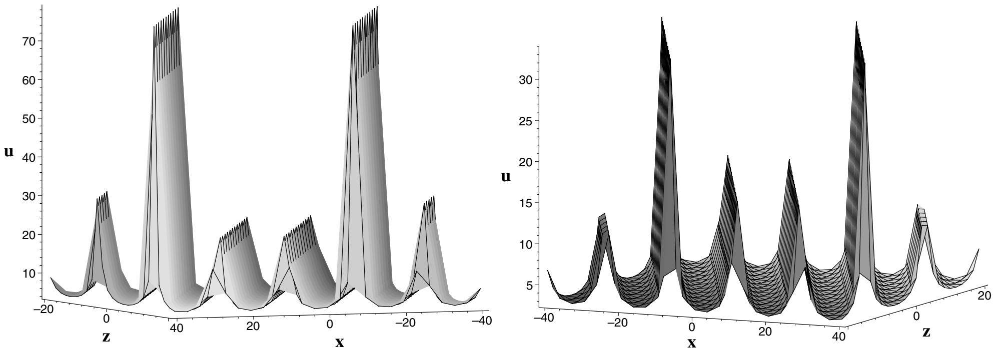

Figure 3. The states of a triangular wave solution u_{12} (equation (3.33), where $N^2 = 0.2, \omega = 2, n = 1, t = 0$) at the x–z plane for (*a*) $Q = 0.1, k = -1$, (*b*) $Q = -0.1, k = 1$.

where it requires that $2\beta^3 - 9\gamma\alpha\beta + 27c\gamma^2 < 0$; so this solution must satisfy the condition

$$5\rho\omega^2 N^2 - 4gn\omega Q < 0, \tag{3.32}$$

which tells us that heating ($Q > 0$) helps to generate upward-propagating solitary waves, but cooling ($Q < 0$) helps to generate downward-propagating solitary waves.

From equations (3.10) and (3.32), we see that heating ($Q > 0$) helps to generate westerly propagating solitary waves if the slope of lines of constant phase is positive, and heating ($Q > 0$) helps to generate easterly propagating solitary waves if the slope of lines of constant phase is negative. However, cooling ($Q < 0$) helps to generate easterly propagating solitary waves if the slope of lines of constant phase is positive, and cooling ($Q < 0$) helps to generate westerly propagating solitary waves if the slope of lines of constant phase is negative.

A triangular solution

$$u_{12} = -\frac{\beta}{3\gamma} + \left(\frac{10(2\beta^3 + 27c\gamma^2 - 9\alpha\beta\gamma)}{27\gamma^3}\sec^2\left(\frac{\sqrt{6\beta^3 + 81c\gamma^2 - 27\alpha\beta\gamma}}{6\gamma}\xi\right)\right)^{1/3}, \tag{3.33}$$

where it requires that $2\beta^3 - 9\gamma\alpha\beta + 27c\gamma^2 > 0$; so this solution must satisfy the condition

$$5\rho\omega^2 N^2 - 4gn\omega Q > 0, \tag{3.34}$$

which tells us that heating ($Q > 0$) helps to generate downward-propagating waves, but cooling ($Q < 0$) helps to generate upward-propagating waves.

From equations (3.10) and (3.34), it is not difficult to see that heating ($Q > 0$) helps to generate easterly propagating waves if the slope of lines of constant phase is positive, and heating ($Q > 0$) helps to generate westerly propagating waves if the slope of lines of constant phase is negative. However, cooling ($Q < 0$) helps to generate westerly propagating waves if the slope of lines of constant phase is positive, and cooling ($Q < 0$) helps to generate easterly propagating waves if the slope of lines of constant phase is negative. The corresponding propagating behaviour is given in figures 3 and 4.

A rational solution

$$u_{13} = -\frac{\beta}{3\gamma} \pm \left(\frac{10}{\gamma\xi^2}\right)^{1/3}, \tag{3.35}$$

where it requires that $5\rho\omega^2 N^2 - 4gn\omega Q = 0$, which shows that the heat term have nearly no effect on the generation and propagation of internal gravity waves.

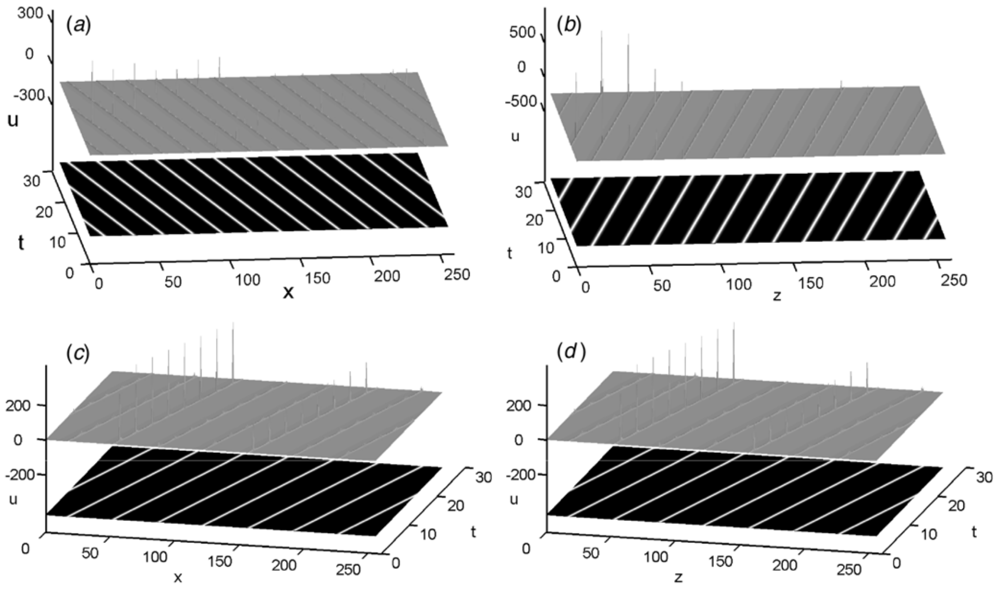

Figure 4. Propagation of a triangular wave solution u_{12} (equation (3.33), where $Q = -0.1$, $N^2 = 0.2, \omega = 2, n = 1$) for (*a*) westerly ($k = -1$), (*b*) upward ($k = 1$), (*c*) easterly ($k = 1$), (*d*) upward ($k = -1$).

When we take $s = 3$ and $r = 5$ in equation (1.3), namely making a transformation $\varphi \to \varphi^{1/3}$ for equation (3.5), then equation (3.5) is reduced to

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = 3\epsilon\sqrt{c_0\varphi^{4/3} + c_1\varphi^{5/3} + c_2\varphi^2 + c_3\varphi^{7/3} + c_4\varphi^{8/3} + c_5\varphi^3}. \tag{3.36}$$

Suppose that $c_0 = c_1 = c_3 = c_4 = 0$; then equation (3.1) also have the same solitary wave solutions as u_{11}, u_{12} and u_{13}.

3.2. The first-order ODE with an eighth-degree nonlinear term

With the aid of Maple9, substitution of equation (3.6) into equation (3.1) shows that the set of algebraic equations possesses the solutions, as detailed in the following sections.

3.2.1. The first kind of solutions with a fifth-degree nonlinear term. The first set of solutions reads

$$\begin{aligned} &c_0 = c_0, \qquad c_1 = c_1, \qquad c_2 = -c - \beta a_0^2 - \gamma a_0^3 - \alpha a_0, \\ &c_3 = -\gamma a_0^2 a_1 - \frac{2\beta a_0 a_1}{3} - \frac{\alpha a_1}{3}, \qquad c_4 = -\frac{\gamma a_0 a_1^2}{2} - \frac{\beta a_1^2}{6}, \qquad c_5 = -\frac{\gamma a_1^3}{10}, \\ &c_6 = 0, \qquad c_7 = 0, \qquad c_8 = 0, \qquad a_0 = a_0, \qquad a_1 = a_1, \qquad a_2 = 0, \end{aligned} \tag{3.37}$$

where $a_1 \neq 0$, a_0, c and c_0 are arbitrary constants. In this case, its solutions have been studied in section 3.1.

3.2.2. The second kind of solutions with an eighth-degree nonlinear term. The second set of solutions reads

$$c_0 = 0, \qquad c_1 = 0, \qquad c_2 = -\frac{c}{4} - \frac{\beta a_0^2}{4} - \frac{\gamma a_0^3}{4} - \frac{\alpha a_0}{4}, \qquad c_3 = 0,$$
$$c_4 = -\frac{\beta a_0 a_2}{6} - \frac{\gamma a_0^2 a_2}{4} - \frac{\alpha a_2}{12}, \qquad c_5 = 0, \qquad c_6 = -\frac{\beta a_2^2}{24} - \frac{\gamma a_0 a_2^2}{8}, \qquad c_7 = 0,$$
$$c_8 = -\frac{\gamma a_2^3}{40}, \qquad a_0 = a_0, \qquad a_1 = 0, \qquad a_2 = a_2, \tag{3.38}$$

where $a_i (i = 0, 1, 2)$ and $c_j (j = 0, 1, \ldots, 8)$ are arbitrary constants.

Then equation (3.6) takes in the form

$$\frac{d\varphi}{d\xi} = \epsilon\sqrt{c_2\varphi^2 + c_4\varphi^4 + c_6\varphi^6 + c_8\varphi^8}. \tag{3.39}$$

Let us make a transformation $\varphi \to \varphi^{1/2}$ for equation (3.39); then equation (3.39) is reduced to

$$\frac{d\varphi}{d\xi} = 2\epsilon\sqrt{c_2\varphi^2 + c_4\varphi^3 + c_6\varphi^4 + c_8\varphi^5}. \tag{3.40}$$

If $\gamma = 0$ and we choose $a_0 = 0$, then $c_2 = -c, c_4 = -\frac{\alpha a_1}{3}, c_6 = -\frac{\beta a_1^2}{6}$, and taking equation (3.40) in the form

$$\frac{d\varphi}{d\xi} = \epsilon\sqrt{c_2\varphi^2 + c_4\varphi^3 + c_6\varphi^4} \tag{3.41}$$

which yields the following solitary wave solutions and triangular periodic wave solutions

$$u_{14} = \frac{6c\alpha\,\mathrm{sech}^2\left(\frac{\sqrt{-c}}{2}\xi\right)}{-2\alpha^2 + 3c\beta - 6c\beta\tanh\left(\frac{\sqrt{-c}\xi}{2}\right) + 3c\beta\tanh^2\left(\frac{\sqrt{-c}\xi}{2}\right)}, \qquad c < 0, \tag{3.42}$$

where it must satisfy $Qng\omega - \rho\omega^2N^2 > 0$, it shows that heating helps to generate upward-propagating waves, but cooling helps to generate downward-propagating waves. If the slope of lines of constant phase is positive, heating helps to generate westerly propagating waves, and if the slope of lines of constant phase is negative, heating helps to generate easterly propagating waves. However, cooling helps to generate easterly propagating waves if the slope of lines of constant phase is positive, and cooling helps to generate westerly propagating waves if the slope of lines of constant phase is negative:

$$u_{15} = \mp\frac{6c\,\mathrm{sech}(\sqrt{-c}\xi)}{\sqrt{(\alpha^2 - 6c\beta)} \pm \alpha\,\mathrm{sech}(\sqrt{-c}\xi)}, \qquad \alpha^2 - 6c\beta > 0, \qquad c < 0 \tag{3.43}$$

$$u_{16} = -\frac{6c\,\mathrm{sech}^2\left(\frac{\sqrt{-c}\xi}{2}\right)}{\pm 2\sqrt{\alpha^2 - 6c\beta} \mp \left(\sqrt{\alpha^2 - 6c\beta} \pm \alpha\right)\mathrm{sech}^2\left(\frac{\sqrt{-c}\xi}{2}\right)}, \qquad \alpha^2 - 6c\beta > 0, \qquad c < 0, \tag{3.44}$$

$$u_{17} = -\frac{6c\,\mathrm{csch}^2\left(\frac{\sqrt{-c}\xi}{2}\right)}{\pm 2\sqrt{\alpha^2 - 6c\beta} \pm \left(\sqrt{\alpha^2 - 6c\beta} \pm \alpha\right)\mathrm{csch}^2\left(\frac{\sqrt{-c}\xi}{2}\right)}, \qquad \alpha^2 - 6c\beta > 0, \qquad c < 0, \tag{3.45}$$

where the solutions (u_{15}, u_{16} and u_{17}) must satisfy the condition

$$2Q^2g^2n^2 + 5QgnN^2\rho\omega - 7N^4\rho^2\omega^2 > 0, \qquad Qng\omega - \rho\omega^2N^2 > 0, \tag{3.46}$$

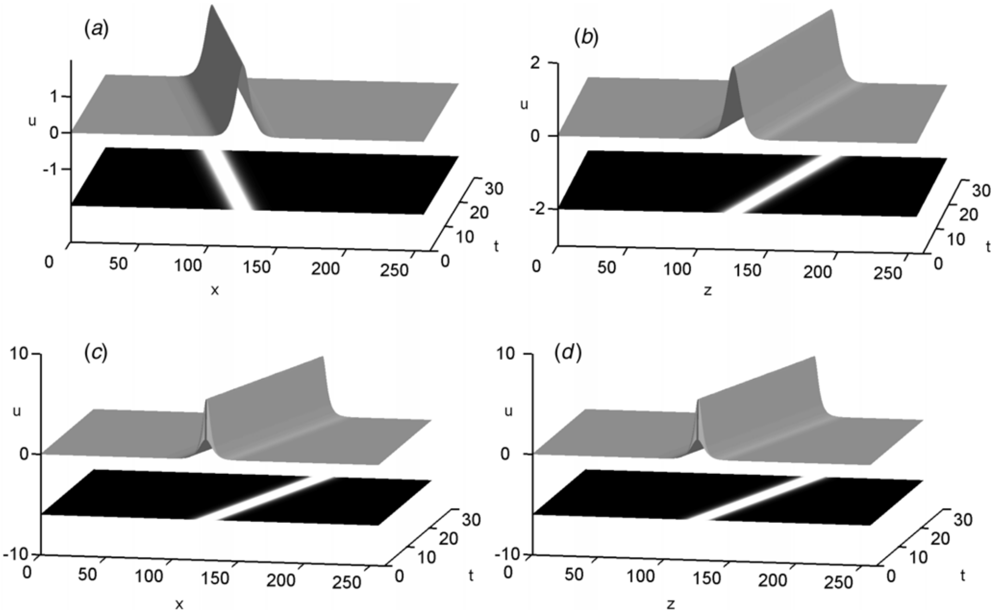

Figure 5. Propagation direction of a solitary wave u_{17} (equation (3.45), where $Q = 0.1, n = 1,$ $N^2 = 0.2, \omega = 2$) for (*a*) westerly ($k = -1$), (*b*) upward ($k = 1$), (*c*) easterly ($k = 1$), (*d*) upward ($k = -1$).

which tells us that heating tends to generate upward-moving waves, but cooling tends to generate downward-moving waves. If the slope of lines of constant phase is positive, heating tends to generate westerly moving waves, and if the slope of lines of constant phase is negative, heating tends to generate easterly moving waves. However, cooling tends to generate easterly moving waves if the slope of lines of constant phase is positive, and cooling helps to generate westerly moving waves if the slope of lines of constant phase is negative. Figure 5 shows the propagating behaviour of the positive branch of u_{17} in qualitative terms

$$u_{18} = -\frac{6c\sec^2\left(\frac{\sqrt{c}\xi}{2}\right)}{\pm 2\sqrt{\alpha^2 - 6c\beta} \mp \left(\sqrt{\alpha^2 - 6c\beta} \pm \alpha\right)\sec^2\left(\frac{\sqrt{c}\xi}{2}\right)}, \qquad \alpha^2 - 6c\beta > 0, \qquad c > 0, \tag{3.47}$$

$$u_{19} = -\frac{6c\csc^2\left(\frac{\sqrt{c}\xi}{2}\right)}{\pm 2\sqrt{\alpha^2 - 6c\beta} \pm \left(\sqrt{\alpha^2 - 6c\beta} \pm \alpha\right)\csc^2\left(\frac{\sqrt{c}\xi}{2}\right)}, \qquad \alpha^2 - 6c\beta > 0, \qquad c > 0, \tag{3.48}$$

where the solutions (u_{18} and u_{19}) must satisfy the condition

$$2Q^2g^2n^2 + 5QgnN^2\rho\omega - 7N^4\rho^2\omega^2 > 0, \qquad Qng\omega - \rho\omega^2N^2 < 0, \tag{3.49}$$

which tells us that under the influence of heating, wave fronts tend to be travelling downward; if the slope of lines of constant phase is positive, wave fronts tend to be travelling in a positive x-direction; and if the slope of lines of constant phase is negative, wave fronts tend to be travelling in a negative x-direction. However, under the influence of cooling, wave fronts tend to be travelling upward; if the slope of lines of constant phase is positive, wave fronts tend to be travelling in a negative x-direction; and if the slope of lines of constant phase is negative,

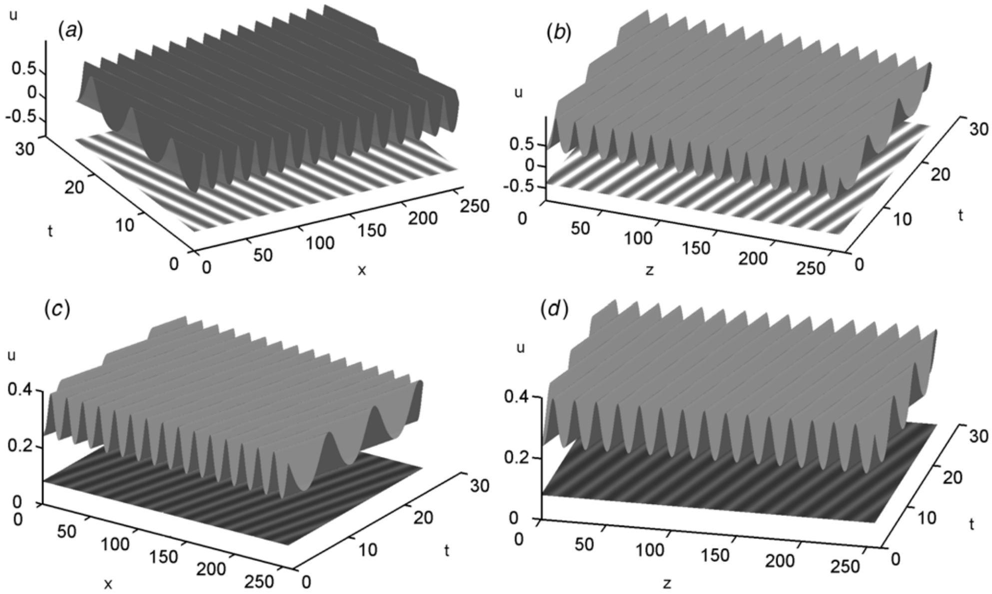

Figure 6. Propagation of wave u_{19} (equation (3.48), where $Q = -0.1$, $n = 1$, $N^2 = 0.2$, $\omega = 2$) for (*a*) westerly ($k = -1$), (*b*) upward ($k = 1$), (*c*) easterly ($k = 1$), (*d*) upward ($k = -1$).

wave fronts tend to be travelling in a positive x-direction. Figure 6 shows the propagating behaviour of the positive branch of u_{19} in qualitative terms.

If $\gamma \neq 0$, when we make a transformation $\varphi \to \varphi^{2/3}$ for equation (3.40), then equation (3.40) is reduced to

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \frac{3}{2}\epsilon\sqrt{c_2\varphi^2 + c_4\varphi^{8/3} + c_6\varphi^{10/3} + c_8\varphi^4}. \tag{3.50}$$

Suppose that $c_4 = c_6 = 0$; then equation (3.1) also have the same solitary wave solutions as u_{11}, u_{12} and u_{13}.

3.2.3. The third kind of solutions with an eighth-degree nonlinear term. The third set of solutions reads

$$c_1 = -a_1\big(-320\beta a_0 a_1^2 a_2^2 + 20\beta a_1^4 a_2 + 60\gamma a_0 a_1^4 a_2 - 3\gamma a_1^6 - 480\gamma a_0^2 a_1^2 a_2^2 - 160 a_1^2 a_2^2\alpha + 1920 c a_2^3 + 1920\gamma a_0^3 a_2^3 + 1920\beta a_0^2 {a_2}^3 + 1920\alpha a_0 a_2^3\big)\big/\big(7680 a_2^4\big),$$

$$c_2 = -\big(320\beta a_0 a_1^2 a_2^2 - 20\beta a_1^4 a_2 - 60\gamma a_0 a_1^4 a_2 + 3\gamma a_1^6 + 480\gamma a_0^2 a_1^2 a_2^2 + 160 a_1^2 a_2^2\alpha + 640 c a_2^3 + 640\gamma a_0^3 a_2^3 + 640\beta a_0^2 a_2^3 + 640\alpha a_0 a_2^3\big)\big/\big(2560 a_2^3\big),$$

$$c_3 = -\frac{a_1\big(320\beta a_0 a_2^2 + 20\beta a_1^2 a_2 + 60\gamma a_0 a_1^2 a_2 - 3\gamma a_1^4 + 480\gamma a_0^2 a_2^2 + 160\alpha a_2^2\big)}{960 a_2^2},$$

$$c_4 = -\frac{64\beta a_0 a_2^2 + 44\beta a_1^2 a_2 + 132\gamma a_0 a_1^2 a_2 + 3\gamma a_1^4 + 96\gamma a_0^2 a_2^2 + 32\alpha a_2^2}{384 a_2},$$

$$c_5 = -\frac{\beta a_1 a_2}{8} - \frac{13\gamma a_1^3}{160} - \frac{3\gamma a_0 a_1 a_2}{8}, \qquad c_6 = -\frac{\beta a_2^2}{24} - \frac{23\gamma a_1^2 a_2}{160} - \frac{\gamma a_0 {a_2}^2}{8},$$

$$c_7 = -\frac{\gamma a_1 a_2^2}{10}, \qquad c_8 = -\frac{\gamma a_2^3}{40}, \qquad c_0 = c_0,$$
$$a_0 = a_0, \qquad a_1 = a_1, \qquad a_2 = a_2, \tag{3.51}$$

where a_i $(i = 0, 1, 2)$ and c_j $(j = 0, 1, \dots, 8)$ are arbitrary constants. Considering the 11 parameters, we present some types of solutions in the following cases.

Case 1. If $c_7 = c_8 = 0$ in equation (3.6), then equation (3.6) becomes

$$\frac{d\varphi}{d\xi} = \epsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4 + c_5\varphi^5 + c_6\varphi^6}. \tag{3.52}$$

By using the condition $c_7 = c_8 = 0$, we can obtain that $a_1 = 0, \gamma = 0, c_1 = c_3 = c_5 = 0$. Suppose that $c_0 = \frac{8c_2^2}{27c_4}$ and $c_6 = \frac{c_4^2}{4c_2}$; then we can obtain $a_0 = \frac{-\alpha\pm\sqrt{3\alpha^2-12c\beta}}{2\beta}$. Then equation (3.1) has a kink profile solution

$$u_{20} = \frac{-\alpha \pm \sqrt{3\alpha^2 - 12c\beta}}{2\beta} \pm \frac{4(\alpha^2 - 4\beta c)\tanh^2\left(\pm\frac{1}{12}\sqrt{\frac{6\alpha^2-24\beta c}{\beta}}\xi\right)}{\beta\sqrt{3\alpha^2 - 12\beta c}\left(3 + \tanh^2\left(\pm\frac{1}{12}\sqrt{\frac{6\alpha^2-24\beta c}{\beta}}\xi\right)\right)} \tag{3.53}$$

and a singular solution

$$u_{21} = \frac{-\alpha \pm \sqrt{3\alpha^2 - 12c\beta}}{2\beta} \pm \frac{4(\alpha^2 - 4\beta c)\coth^2\left(\pm\frac{1}{12}\sqrt{\frac{6\alpha^2-24\beta c}{\beta}}\xi\right)}{\beta\sqrt{3\alpha^2 - 12\beta c}\left(3 + \coth^2\left(\pm\frac{1}{12}\sqrt{\frac{6\alpha^2-24\beta c}{\beta}}\xi\right)\right)}, \tag{3.54}$$

where $\alpha^2 - 4\beta c > 0$ and $\beta > 0$ in equations (3.53) and (3.54); so the solutions must satisfy condition (3.10), where the wave propagation directions have been studied in detail in section (3.1).

A triangular periodic solution

$$u_{22} = \frac{-\alpha \pm \sqrt{3\alpha^2 - 12c\beta}}{2\beta} \pm \frac{4(\alpha^2 - 4\beta c)\tan^2\left(\pm\frac{1}{12}\sqrt{-\frac{6\alpha^2-24\beta c}{\beta}}\xi\right)}{\beta\sqrt{3\alpha^2 - 12\beta c}\left(3 - \tan^2\left(\pm\frac{1}{12}\sqrt{-\frac{6\alpha^2-24\beta c}{\beta}}\xi\right)\right)} \tag{3.55}$$

and a singular triangular periodic solution

$$u_{23} = \frac{-\alpha \pm \sqrt{3\alpha^2 - 12c\beta}}{2\beta} \pm \frac{4(\alpha^2 - 4\beta c)\cot^2\left(\pm\frac{1}{12}\sqrt{-\frac{6\alpha^2-24\beta c}{\beta}}\xi\right)}{\beta\sqrt{3\alpha^2 - 12\beta c}\left(3 - \cot^2\left(\pm\frac{1}{12}\sqrt{-\frac{6\alpha^2-24\beta c}{\beta}}\xi\right)\right)}, \tag{3.56}$$

where $\alpha^2 - 4\beta c > 0$ and $\beta < 0$ in equations (3.55) and (3.56) respectively; so the solutions must satisfy the condition

$$Q^2g^2n^2 + QgnN^2\rho\omega - 2N^4\rho^2\omega^2 > 0, \qquad N^2 < 0, \tag{3.57}$$

which tells us that in the unstable atmosphere ($N^2 < 0$), heating ($Q > 0$) helps to generate downward-propagating waves, but cooling ($Q < 0$) helps to generate upward-propagating waves.

From equations (3.10) and (3.57), it is not difficult to see that in the unstable atmosphere ($N^2 < 0$), heating ($Q > 0$) helps to generate easterly propagating waves if the slope of lines of constant phase is positive, and heating ($Q > 0$) helps to generate westerly propagating waves if the slope of lines of constant phase is negative.

However, in the unstable atmosphere ($N^2 < 0$), cooling ($Q < 0$) helps to generate westerly propagating waves if the slope of lines of constant phase is positive, and cooling

Table 1. The wave propagation direction from the local heating.

Solution type	Stability	Upward	Downward	Westerly	Easterly	ODE type
u_1, u_4	$N^2 > 0$	$Q > 0$	$Q < 0$	$Q \cdot Sp > 0$	$Q \cdot Sp < 0$	ODE5, ODE8
u_2, u_5	$N^2 > 0$	$Q < 0$	$Q > 0$	$Q \cdot Sp < 0$	$Q \cdot Sp > 0$	ODE5, ODE8
u_{11}	–	$Q > 0$	$Q < 0$	$Q \cdot Sp > 0$	$Q \cdot Sp < 0$	ODE5, ODE8
u_{12}	–	$Q < 0$	$Q > 0$	$Q \cdot Sp < 0$	$Q \cdot Sp > 0$	ODE5, ODE8
$u_{14}, u_{15}, u_{16}, u_{17}$	–	$Q > 0$	$Q < 0$	$Q \cdot Sp > 0$	$Q \cdot Sp < 0$	ODE8
u_{18}, u_{19}	–	$Q < 0$	$Q > 0$–	$Q \cdot Sp < 0$	$Q \cdot Sp > 0$	ODE8
u_{22}, u_{23}	$N^2 < 0$	$Q < 0$	$Q > 0$	$Q \cdot Sp < 0$	$Q \cdot Sp > 0$	ODE8
$u_{20}, u_{21}, u_{24}, u_{25}$	$N^2 > 0$	$Q > 0$	$Q < 0$	$Q \cdot Sp > 0$	$Q \cdot Sp < 0$	ODE8

($Q < 0$) helps to generate easterly propagating waves if the slope of lines of constant phase is negative.

If we suppose that $h_0 = 0$ and $c_6 = \frac{c_4^2}{4c_2}$, then equation (3.1) has a kink profile solution

$$u_{24} = \frac{-\alpha \pm \sqrt{3\alpha^2 - 12c\beta}}{2\beta} \pm \frac{3(\alpha^2 - 4\beta c)}{2\beta\sqrt{3\alpha^2 - 12\beta c}} \left(1 + 3\tanh\left(\pm\sqrt{\frac{2\alpha^2 - 8\beta c}{\beta}}\xi\right)\right) \tag{3.58}$$

and a singular solution

$$u_{25} = \frac{-\alpha \pm \sqrt{3\alpha^2 - 12c\beta}}{2\beta} \pm \frac{3(\alpha^2 - 4\beta c)}{2\beta\sqrt{3\alpha^2 - 12\beta c}} \left(1 + 3\coth\left(\pm\sqrt{\frac{2\alpha^2 - 8\beta c}{\beta}}\xi\right)\right), \tag{3.59}$$

where $\alpha^2 - 4\beta c > 0$ and $\beta > 0$ in equations (3.58) and (3.59) respectively; so the solutions must satisfy condition (3.10), where the wave propagation directions have been studied in detail in section (3.1).

Case 2. If we take $s = 5$ and $r = 8$ in equation (1.3), namely making a transformation $\varphi \to \varphi^{1/2}$ for equation (3.6), then equation (3.6) becomes

$$\frac{d\varphi}{d\xi} = 2\epsilon\sqrt{c_0\varphi + c_1\varphi^{3/2} + c_2\varphi^2 + c_3\varphi^{5/2} + c_4\varphi^3 + c_5\varphi^{7/2} + c_6\varphi^4 + c_7\varphi^{9/2} + c_8\varphi^5}. \tag{3.60}$$

Supposing that $c_1 = c_3 = c_5 = c_7 = 0$, equation (3.1) also has the same solitary wave solutions as u_{11}, u_{12} and u_{13}.

4. Summary and discussions

We have extended Fan's direct and unified algebraic method with symbolic computation to the case when $r > 4$ by introducing a new transformation and successfully applied this to investigate internal gravity waves in the atmosphere. The solutions of a first-order nonlinear ordinary differential equation with a higher degree nonlinear term, such as with a fifth-degree nonlinear term (ODE5) and an eighth degree nonlinear term (ODE8), are obtained, which included periodic wave solutions associated with the Jacobin elliptic function and the bell-shaped and kink profile solitary wave solutions. Most importantly, the propagation and generation of gravity waves are affected by the local heating conditions; the results are listed in table 1, where $Sp = \frac{dz}{dx}$ denotes the slope of lines of constant phase in the x–z plane, $Q \cdot Sp > 0$ denotes $Q > 0$ and $Sp > 0$ or $Q < 0$ and $Sp < 0$, and $Q \cdot Sp < 0$ denotes $Q > 0$ and $Sp < 0$ or $Q < 0$ and $Sp > 0$.

Except those considered in this paper, the proposed method is of great significance in many fields in physics, mechanics, atmosphere and ocean, etc, and it is also readily applicable to a large variety of other nonlinear evolution equations such as the generalized coupled Hirota–Satsuma, coupled Schrödinger–KdV, (2+1)-dimensional dispersive long wave, (2+1)-dimensional Davey–Stewartson equations, the (3+1)-dimensional Jimbo–Miwa equation, etc. The details for these cases will be investigated in our future work.

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